Notes on Parity Misconceptions

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1 Introduction

The Rubik's Revenge (4x4x4) is notorious for its so-called "parity" states encountered during solving, which can be most easily described as impossible states of the Rubik's Cube (3x3x3). However, few speedsolvers are aware of the mathematical basis behind their appearance (i.e., why these "parity" states appear on the 4x4x4 but not the 3x3x3) or what the term "parity" actually means. This write-up is intended to be a "mathematically accurate" explanation of twisty puzzle parity that avoids excessive jargon and symbols that can hinder understanding by people unfamiliar with abstract algebra/group theory. It assumes that readers are proficient with 3x3x3 solution methods utilizing Orient Last Layer (OLL) and Permute Last Layer (PLL) algorithms, as well as the 4x4x4 reduction method. All puzzle visualizations are drawn using the Western color scheme in the standard scrambling orientation, where the U face is white and the F face is green, unless indicated otherwise.

2 Mathematical Definitions of Parity

In math, parity refers to whether an integer is even or odd. Obviously, fraction and decimal numbers cannot have parity properties. Even numbers are divisible by 2, and odd numbers are not. Even numbers have even parity, which can be alternatively expressed as a parity of 0. Similarly, odd numbers have odd parity, which can be alternatively expressed as a parity of 1.

Example:
-10, 0, 2, and 46 are divisible by 2 $\rightarrow$ even numbers, even parity, parity of 0
-27, 1, and 55 are not divisible by 2 $\rightarrow$ odd numbers, odd parity, parity of 1

Suppose a set "S" exists with the elements A, B, and C. Now, suppose these elements are placed in a specific order, or a permutation, like so: (A B C). (A B C) is the identity permutation of set "S," which can also be thought of as its default order. Of course, other than the identity permutation, many additional permutations of set "S" also exist. Each of these permutations can be created through a series of transpositions, or swaps of 2 elements at a time, starting from the identity permutation.

Example:
S = {A, B, C}

For example, swapping A and B in (A B C) results in a new permutation: (B A C). (B A C) exhibits odd parity because it was created using 1 swap of 2 elements. With reference to the identity permutation, odd permutations are always created/restored using an odd number of swaps. Next, suppose A and C in (B A C) are swapped, resulting in (B C A). (B C A) exhibits even parity because it was created using 2 swaps. With reference to the identity permutation, even permutations are always created/restored using an even number of swaps.

Example:
(A B C) (identity, 0 swaps = even) $\rightarrow$ (B A C) (1 swap = odd) $\rightarrow$ (B C A) (2 swaps = even)

It is impossible for a permutation created using an even number of swaps to be brought back to the identity permutation using an odd number of swaps. The same is true for the opposite: a permutation created using an odd number of swaps can never be brought back to the identity permutation using an even number of swaps.

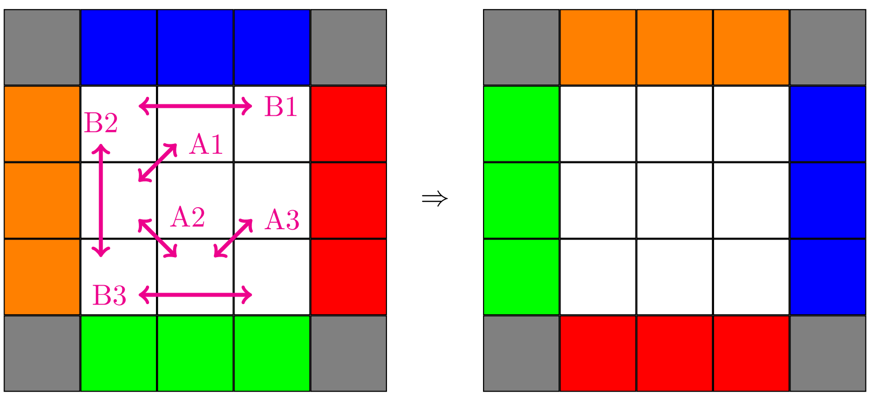
3 Parity Applications for Twisty Puzzles

The same concepts can be applied to the 3x3x3 and, for that matter, all twisty puzzles. A puzzle can be viewed as a set with each piece corresponding to an element in that set. The solved state of the puzzle is its identity permutation. Then, comparing all legal scrambled states to the solved state allows them to be designated as even or odd permutations of the puzzle in question. Even puzzle states are created by applying an even number of swaps to the solved puzzle state, and odd puzzle states are created by applying an odd number of swaps to the solved puzzle state. The solved puzzle state can be viewed as having 0 swaps applied, making it an even puzzle state itself.

4 3x3x3 Overall Parity

For the 3x3x3, only even permutations are legal. This is because the simplest legal move on the 3x3x3 is a quarter face turn, which performs a 4-cycle of edges and a 4-cycle of corners. Each 4-cycle consists of 3 separate piece swaps, summing to a total of 6 piece swaps.

Example:
[U] performs a clockwise 4-cycle of edges (A) and a clockwise 4-cycle of corners (B).



Even numbers combined with even numbers always result in even numbers. Since the simplest legal move performs an even number of swaps, any combination of legal moves on the 3x3x3 will also perform an even number of swaps. Thus, given that the solved puzzle state is even, any scrambled puzzle state reached using legal moves will also be even.

5 3x3x3 Piece-Specific Parity

Recall that the simplest legal move on the 3x3x3 performs 3 swaps of edges and 3 swaps of corners. This means that an odd permutation of edges is possible if coupled with an odd permutation of corners, and vice versa. In essence, edges and corners can be permuted evenly and separately without restrictions, but they can be permuted oddly only in conjunction with each other.

In summary of 3x3x3 permutation legality:
Even permutation overall: Allowed
Odd permutation overall: Forbidden
Even permutation of edges: Allowed
Odd permutation of edges: Restricted—Must be linked to an odd permutation of corners

Even permutation of corners: Allowed
Odd permutation of corners: Restricted—Must be linked to an odd permutation of edges

These rules are observed when considering the PLL algorithms, of which there are 21. Each can be exclusively placed into 1 of 4 possible classifications as shown below:

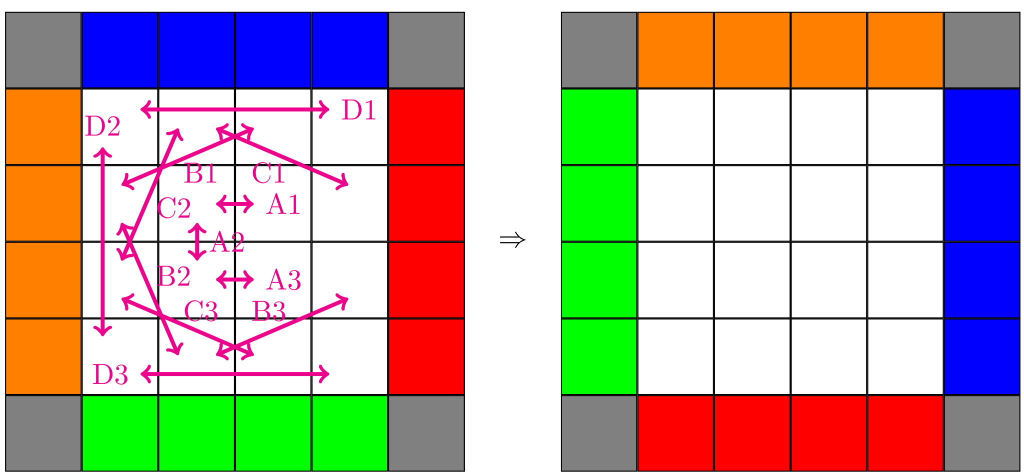
Even permutations of edges only: H, Ua/Ub, Z
Even permutations of corners only: Aa/Ab, E
Odd permutations of both edges and corners: F, Ja/Jb, Na/Nb, Ra/Rb, T, V, Y
Even permutations of both edges and corners: Ga/Gb/Gc/Gd

Again, it is impossible to create an odd permutation of either piece type without creating an odd permutation of the other type.

6 4x4x4 Overall Parity

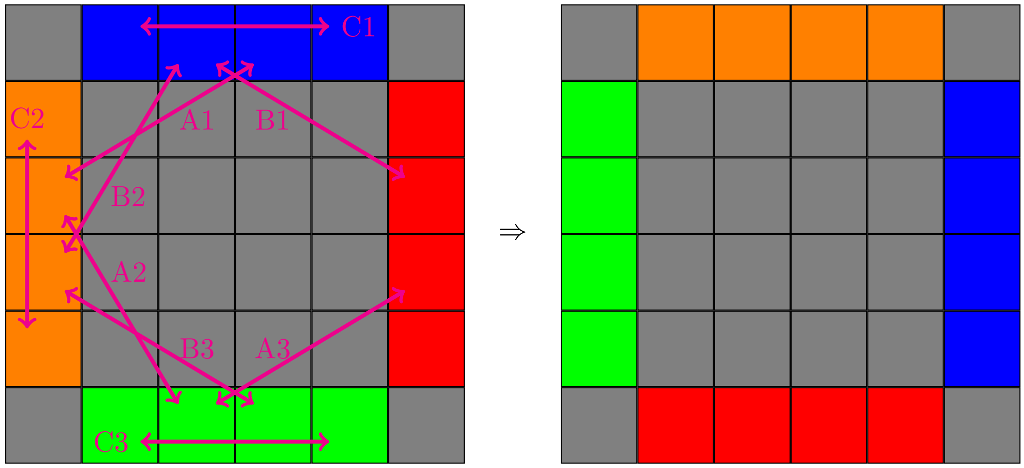
In contrast to the 3x3x3, both even and odd permutations of the 4x4x4 are legal. This is due to the 4x4x4 permitting 2 types of simplest legal moves: quarter face (outer layer) turns and quarter slice (inner layer) turns. Quarter face turns perform a 4-cycle of centers, two 4-cycles of edges, and a 4-cycle of corners. Each 4-cycle consists of 3 separate piece swaps, summing to a total of 12 piece swaps. Thus, all face turns preserve the parity of the current puzzle state. Even 4x4x4 states subjected to any combination of face turns will always remain even.

Example:
[U] performs a clockwise 4-cycle of centers (A), two clockwise 4-cycles of edges (B, C), and a clockwise 4-cycle of corners (D).



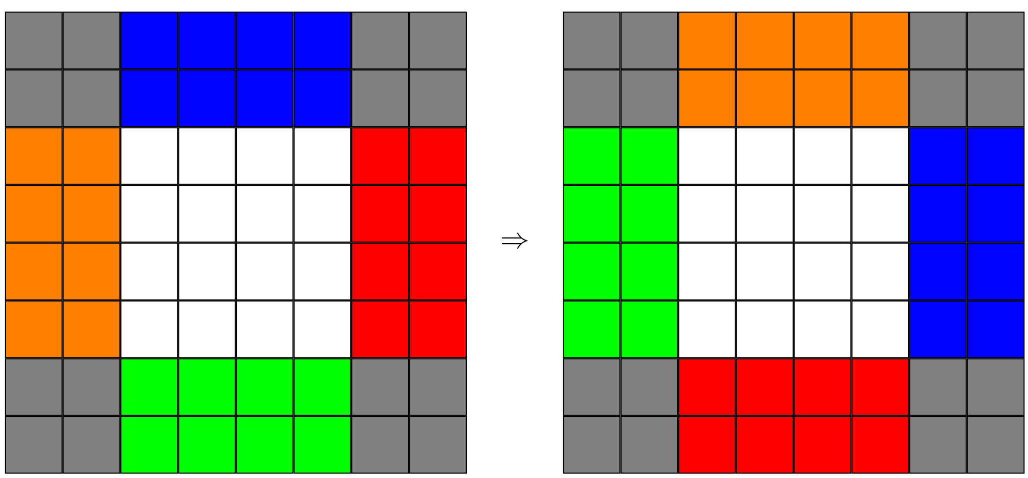
Quarter slice turns perform two 4-cycles of centers and a 4-cycle of edges, for a total of 9 piece swaps. Since it is now possible for a legal move to perform an odd number of swaps, odd puzzle states can be reached on the 4x4x4. Parity is preserved with any combination of moves containing an even number of quarter slice turns, while parity is switched with any combination of moves containing an odd number of quarter slice turns.

Example:
[u] performs two clockwise 4-cycles of centers (A, B) and a clockwise 4-cycle of edges (C).



Note that a quarter wide turn simply combines a quarter face turn and a quarter slice turn, resulting in an odd number of swaps. As a result, outer layers are affected by face and wide turns, while inner layers are affected by slice and wide turns. Expanding a previous statement, parity is preserved with any combination of face, slice, and wide moves containing an even number of quarter inner layer turns. Parity is switched with any combination of face, slice, and wide moves containing an odd number of quarter inner layer turns.

Example:
[Uw] = [U u]



7 4x4x4 Piece-Specific Parity

Recall that quarter face turns perform 3 swaps of centers, 6 swaps of edges, and 3 swaps of corners. From this, it can be determined that odd permutations of centers must be coupled with odd permutations of corners. Even permutations of all 3 types of pieces are allowed without restrictions. Next, recall that quarter slice turns perform 3 swaps of edges and 6 swaps of centers. This demonstrates that odd permutations of edges are also allowed without restrictions.

In summary of 4x4x4 permutation legality:
Even permutation overall: Allowed

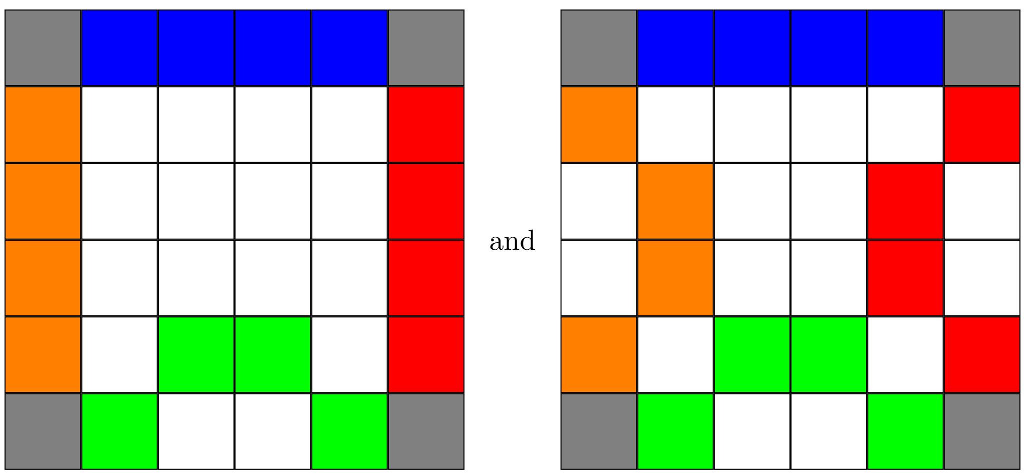
Odd permutation overall: Allowed
Even permutation of centers: Allowed
Odd permutation of centers: Restricted—Must be linked to an odd permutation of corners
Even permutation of edges: Allowed
Odd permutation of edges: Allowed
Even permutation of corners: Allowed
Odd permutation of corners: Restricted—Must be linked to an odd permutation of centers

8 Common Meanings of "Parity" in Twisty Puzzle Contexts

The mathematical concepts behind twisty puzzle solving are quite complex, and over time, the term "parity" has become a classification of any state possible on higher-order regular cuboids that are not possible on the 3x3x3. Most prominently, on the 4x4x4, 2 types of "parity" are possible: OLL Parity and PLL Parity. They are so named due to the popularity of the reduction method for solving the 4x4x4, which, as implied by its name, "reduces" the 4x4x4 to a 3x3x3 during the solution process.

OLL Parity is any 4x4x4 state which presents with a seemingly illegal OLL pattern during the 3x3x3 stage. In its most solved/restored form, the OLL Parity state presents as a fully-solved 4x4x4, except for 1 flipped edge pair.

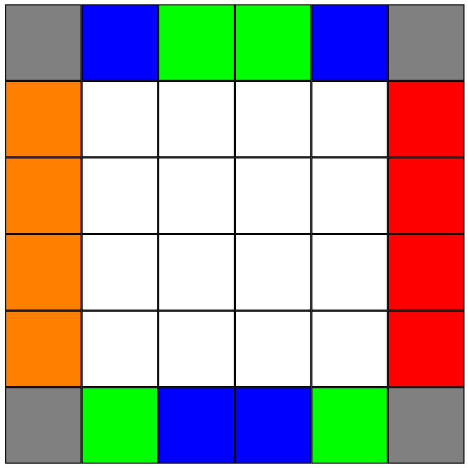
Example:
2 possible 4x4x4 states exhibiting OLL Parity



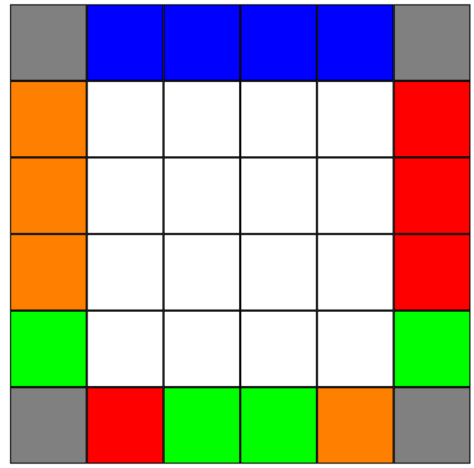
Left: OLL Parity involving the UF edges
Right: OLL Parity involving the UL, UF, and UR edges

PLL Parity is any 4x4x4 state which presents with a seemingly illegal PLL pattern during the 3x3x3 stage. In its most solved/restored form, the PLL Parity state presents as a fully-solved 4x4x4, except for 2 swapped edge pairs (henceforth referred to as PLL Parity Edges) or 2 swapped corners (henceforth referred to as PLL Parity Corners).

Example:
2 possible 4x4x4 states exhibiting PLL Parity



and



Left: PLL Parity Edges involving the UF and UB edges
Right: PLL Parity Corners involving the UFL and UFR corners

From this point, the term "parity" refers to the colloquial speedsolving definition when appearing as part of the "OLL Parity" and "PLL Parity" terms, as well as when found in quotation marks. Unmarked uses of the term "parity" refer to its mathematical definition.

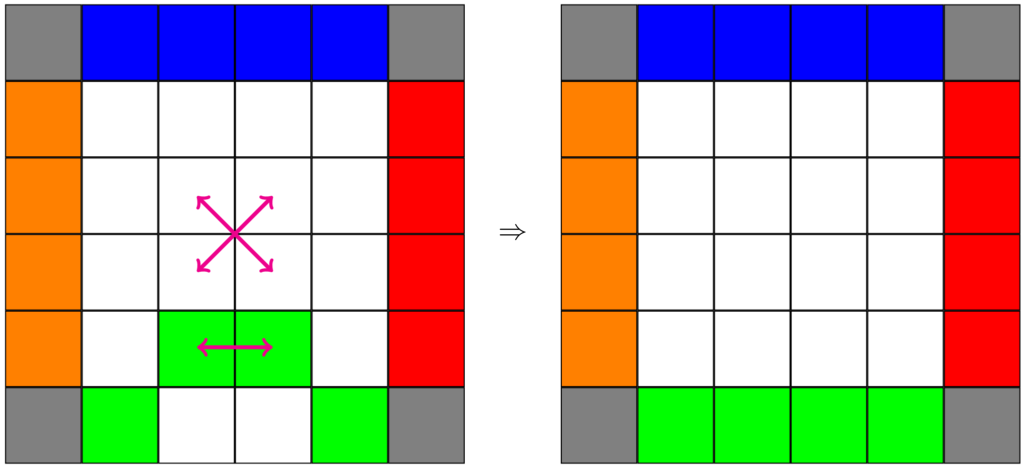
9 OLL Parity Analysis

OLL Parity states are strictly odd permutations. This is because 4x4x4 edges have asymmetrical constructions and can only be oriented 1 way when fixed to any position. OLL Parity states are resolved by swapping the 2 affected edge pieces, which is an odd number of swaps, and odd permutations of edges are allowed without restrictions. Accordingly, all OLL Parity algorithms contain an odd number of quarter inner layer turns. Consider the standard OLL Parity algorithm:

OLL-P1: r2 B2 U2 l U2 r' U2 r U2 F2 r F2 l' B2 r2

OLL-P1 contains 9 quarter inner layer turns, indicating that it converts the 4x4x4 between odd and even puzzle states each time it is executed. Specifically, it swaps the 2 edges involved in the OLL Parity state (UF edges), in addition to performing 2 diagonal swaps of the U centers. This totals to 3 swaps, an odd number which can also be used to argue that OLL-P1 switches the 4x4x4 between odd and even puzzle states.

OLL-P1 transpositions:

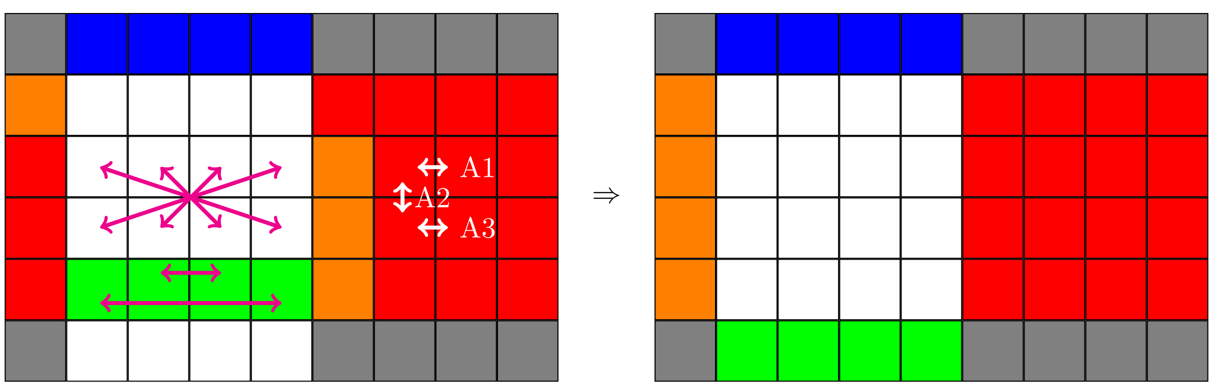


OLL-P1 certainly is a popular algorithm, but it uses slice turns to affect the inner layers. Wide turns are typically faster than slice turns when it is necessary to manipulate specific edges such as in OLL Parity states, leading to the relevance of the following alternative OLL Parity algorithm:

OLL-P2: Rw2 B2 U2 Lw U2 Rw' U2 Rw U2 F2 Rw F2 Lw' B2 Rw2

As with OLL-P1, OLL-P2 contains 9 quarter inner layer turns, which can be reasoned by the fact that OLL-P2 simply converts all OLL-P1 slice turns to wide turns. However, OLL-P2 performs 9 swaps, as opposed to the 3 swaps by OLL-P1. Specifically, OLL-P2 will perform 1 swap of the OLL Parity-affected edges (UF edges), 1 swap of 2 adjacent corners (UFL and UFR corners), 2 diagonal swaps of 4 adjacent edges (UL and UR edges), 2 diagonal swaps of the U centers, and 3 swaps of the R centers (in the form of a clockwise 4-cycle). Note that OLL-P2 also reorients the UFL and UFR corners.

OLL-P2 transpositions:

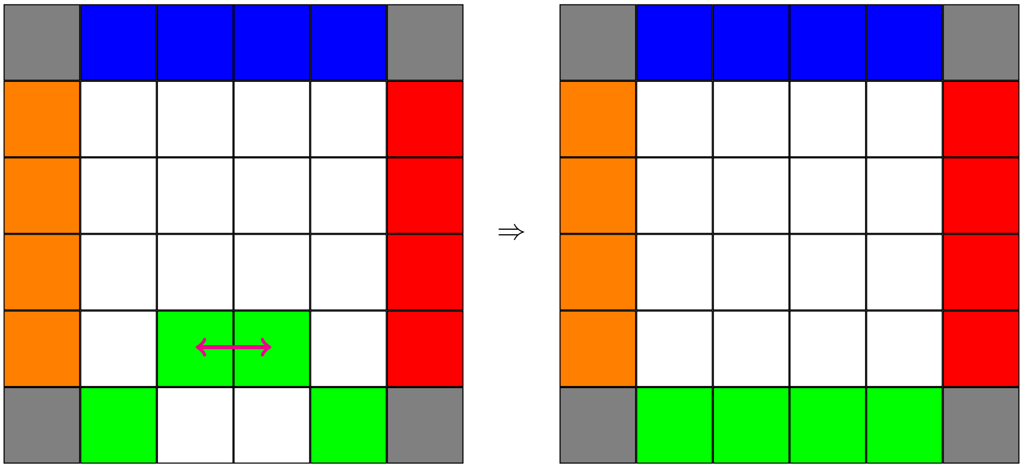


Recall that odd permutations of edges are allowed without restrictions on the 4x4x4. Thus, it is possible to swap the OLL Parity-involved edges without affecting any other pieces. An algorithm that accomplishes this is the following:

OLL-P3: r' U' u r U' r U r u' r' u r U r U' r Uw' r' U2

OLL-P3 performs 1 swap of the OLL Parity-affected edges (UF edges) via 13 quarter inner layer turns.

OLL-P3 transpositions:



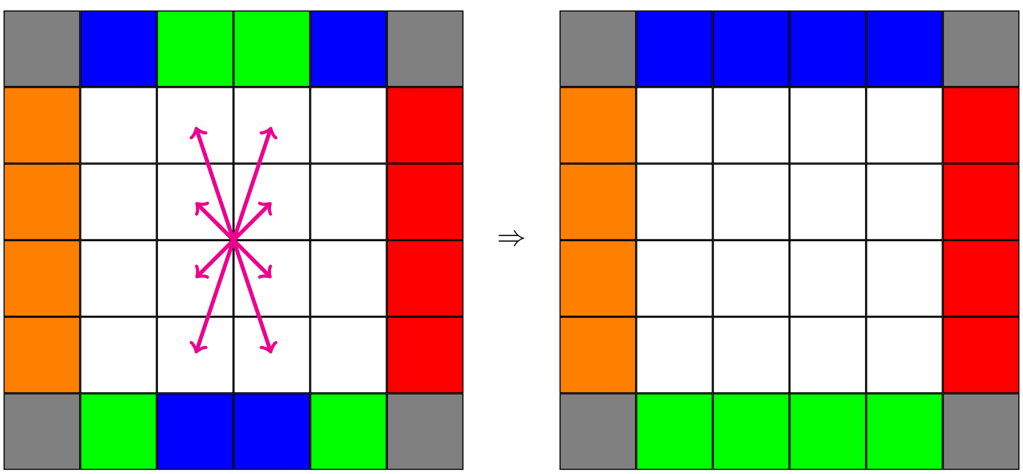
10 PLL Parity Analysis

PLL Parity states are strictly even permutations. Consider PLL Parity Edges states, which consist of 2 swaps of 4 individual edges. Since this is an even number of swaps, PLL Parity Edges states are even. The standard PLL Parity Edges algorithm is as follows:

PLL-P1: r2 U2 r2 Uw2 r2 u2

Like OLL-P1, PLL-P1 will swap pieces other than the parity-involved edges. Namely, PLL-P1 performs 2 diagonal swaps of the UF and UB edges and 2 diagonal swaps of the U centers via 10 quarter inner layer turns.

PLL-P1 transpositions:

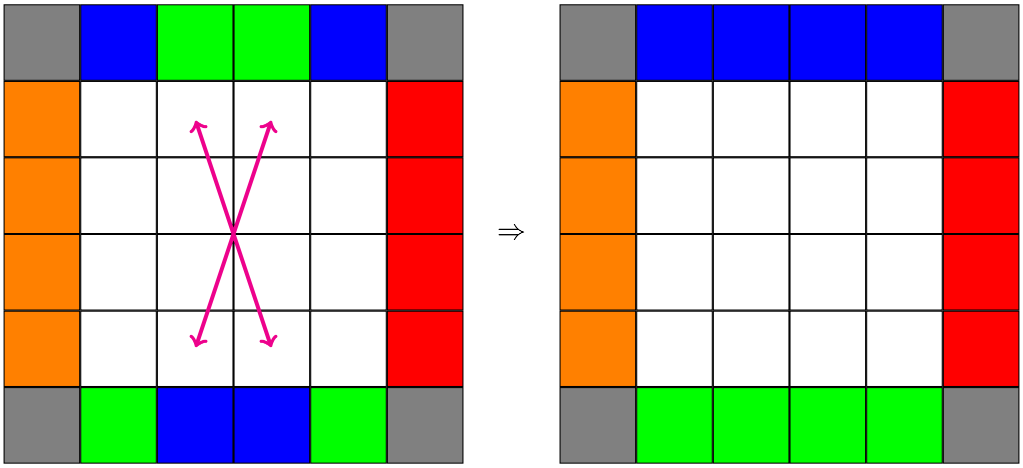


Another algorithm that resolves PLL Parity Edges states without disturbing other pieces is the following:

PLL-P2: r2 U2 r U2 r2 U2 r2 U2 r U2 r2 U2

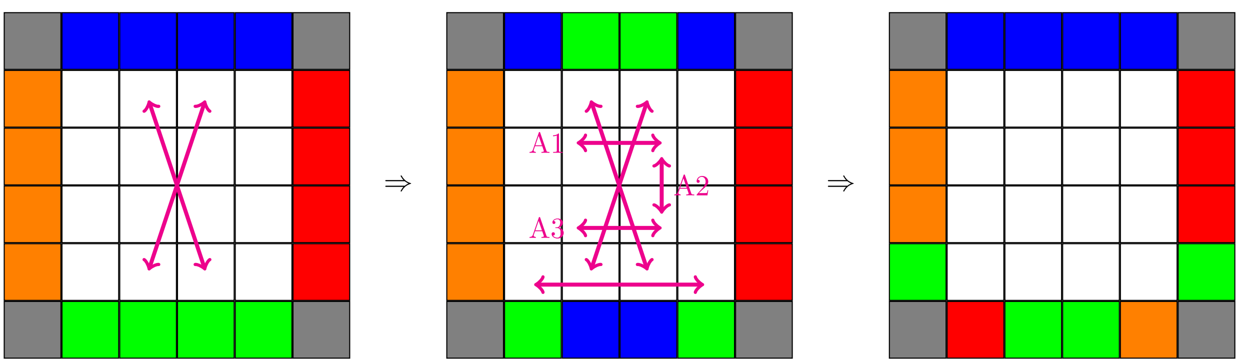
PLL-P2 swaps the UF and UB edges, only, via 10 quarter inner layer turns.

PLL-P2 transpositions:



Now, consider the PLL Parity Corners state, consisting of 1 swap of 2 individual corners. Although this appears to be an odd puzzle state, recall that odd permutations of corners must be linked to odd permutations of centers on the 4x4x4, summing to even puzzle states. These odd permutations of centers, however, are normally not observable due to the indistinguishability of same-color centers on standard 4x4x4s. Nonetheless, all PLL Parity algorithms contain an even number of quarter inner layer turns.

Example: PLL-P2 performs 2 swaps of the UF and UB edges. Then, a T-permutation translated to the F face reverses the UF and UB edges swaps. Additionally, it performs 1 swap of the UFL and UFR corners and 3 swaps of the U centers (in the form of a counterclockwise 4-cycle). The final puzzle state exhibits PLL Parity Corners, has even parity, and is reached using 8 swaps (or 4 swaps if the UF and UB edges swaps are ignored).



11 Supercube Considerations

Standard 4x4x4 cubes are manufactured with distinct solid colors on each face, leading centers of the same color to be indistinguishable. As a result, the most commonly-used OLL Parity and PLL Parity algorithms swap centers for brevity and convenience, in addition to resolving observable "parity" states. Such algorithms, however, would be unusable on 4x4x4 "supercubes," which are colored/marked such that every piece is distinguishable. To solve supercubes, it is necessary to have "supercube-safe" algorithms that perform specific swaps (including those required by OLL Parity and PLL Parity states) without disturbing the centers. OLL-P3 is an example of a supercube-safe OLL Parity algorithm, and PLL-P2 is an example of a supercube-safe PLL Parity Edges algorithm. By definition, PLL Parity Corners algorithms are not supercube-safe. They must oddly swap centers as well as the parity-involved corners.

12 Resources

I highly recommend the resources listed below which were instrumental in my understanding of twisty puzzle parity and the completion of this write-up. They discuss group theory and parity in much greater technical detail.

Leonard F. Bakker – "Math 371 Lecture #29: The Symmetric and Alternating Groups"
https://math.byu.edu/∼bakker/Math371/LectureNotes/M371Lec29.pdf
Note: The tilde rendered by LaTeX may result in an error when the URL is copied. Please type the tilde manually to access this resource.

Tadeáš Miler and Christopher Mowla – "Parity problem or solutions to unsolvable situations"
https://hlavolam.maweb.eu/parity-problem

Mogens Esrom Larsen – "Rubik's Revenge: The Group Theoretical Solution"
https://www.maa.org/sites/default/files/pdf/pubs/Rubiks8.pdf

Speedsolving.com Wiki – "4x4x4 parity algorithms"
https://www.speedsolving.com/wiki/index.php?title=4x4x4_parity_algorithms